

ELLIPTIC YANG–MILLS FLOW THEORY

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Dedicated to Professor Dietmar A. Salamon on the occasion of his 60th birthday

PREVIEW

ABSTRACT. We lay the foundations of a Morse homology on the space of connections on a principal G -bundle over a compact manifold Y , based on a newly defined gauge-invariant functional \mathcal{J} . While the critical points of \mathcal{J} correspond to Yang–Mills connections on P , its L^2 -gradient gives rise to a novel system of elliptic equations. This contrasts previous approaches to a study of the Yang–Mills functional via a gradient flow. We work out in the two-dimensional case the analytical foundations of a Yang–Mills homology based on the functional \mathcal{J} . An application of this theory is given for three-dimensional products $Y = \Sigma \times S^1$.

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1. INTRODUCTION

During the last decades many authors, in order to understand Morse theoretical properties of the Yang–Mills functional

$$\mathcal{YM}: \mathcal{A}(P) \rightarrow \mathbb{R}, \quad \mathcal{YM}(A) = \frac{1}{2} \int_Y \langle F_A \wedge *F_A \rangle$$

on a principal G -bundle P over a compact manifold Y , studied its L^2 gradient flow

$$(1) \quad \partial_s A + d_A^* F_A = 0.$$

This approach was introduced by Atiyah and Bott (cf. [3]) and used for example by Donaldson (cf. [5]) to prove a generalized version of the Narasimhan–Seshadri theorem. Analytical properties of the solutions of (1) for principal bundles over a 2- or 3-dimensional base manifold, or over base manifolds with a symmetry of codimension 3, were proven by Råde (cf. [15], [16]) and Davis (cf. [4]). A main consideration about this flow is the following one. Since an infinite dimensional group of symmetries, the group $\mathcal{G}(P)$ of gauge transformations, acts on the space $\mathcal{A}(P)$ of connections, equation (1) is not parabolic in contrary to the heat flow on manifolds. Its linearisation, augmented by a gauge fixing condition, splits into a parabolic and an elliptic operator (cf. [4], [12], [22]). Moreover, it is not known whether the flow satisfies the Morse–Smale transversality property and it is therefore in the general case an open question whether a Morse homology, based on the Yang–Mills functional and its L^2 -flow, can be defined.

In this context we present a novel approach to Yang–Mills theory and define a new Morse homology in the following way. We consider a principal G -bundle $P \rightarrow Y$, G a compact Lie group with Lie algebra \mathfrak{g} , over a n -dimensional base manifold Y together with the space $X := \mathcal{A}(P) \times \Omega^{n-2}(Y, \text{ad } P)$ defined as the product between the space of \mathfrak{g} -valued connections on P and $\text{ad}(P)$ -valued $(n-2)$ -forms. We fix a Riemannian metric g on Y and let dvol_Y denote its volume form. Then we consider the energy functional

$$(2) \quad \mathcal{J}: X \rightarrow \mathbb{R}, \quad \mathcal{J}(A, \omega) := \int_Y \left(\langle F_A, *\omega \rangle - \frac{1}{2} |\omega|^2 \right) \text{dvol}_Y.$$

The critical points of this functional satisfy the system of equations

$$(3) \quad *F_A - \omega = 0, \quad d_A \omega = 0$$

and thus, in this case, A is a Yang–Mills connection and $*\omega$ is its curvature. The most interesting aspect are the L^2 gradient flow equations

$$(4) \quad \partial_s A + (-1)^{n+1} * d_A \omega = 0, \quad \partial_s \omega + (F_A - \omega) = 0,$$

which are of first order and for $n = 2$ define a nice elliptic problem. Using the gradient flow equations (4), we may then define a new elliptic Yang–Mills homology.

The plan of the paper is as follows. In Section 2 we discuss the elliptic Yang–Mills equations and establish in the following the main properties in order to define elliptic Yang–Mills homology in the case $n = 2$ (which we introduce in Section 6). We in particular show that the linearization of the gradient flow equations give rise to a Fredholm operator and determine its index. The main difficulty here is that the functional J is neither bounded from below nor from above so that the number of eigenvalue crossings of the resulting spectral flow cannot be read off from the index of the Hessian at limiting critical points of \mathcal{J} . To overcome this problem we relate this spectral flow to that of a further family of elliptic operators, the numbers of negative eigenvalues of their limits as $s \rightarrow \pm\infty$ being finite and equal to the index of the limiting Yang–Mills connections. Exponential decay of finite energy solutions of (4) towards critical points is shown in Section 5. A further result concerns compactness up to gauge transformations and convergence to broken trajectories of the moduli spaces of solutions of (4) of uniformly bounded energy. A transversality result for linearized sections is omitted in this version of the paper. It can be obtained using standard holonomy perturbations as e.g. employed in [21].

As an application of the elliptic Yang–Mills homology presented here we consider in Section 7 three dimensional product manifolds $Y = \Sigma \times S^1$. We relate the new invariants obtained in this case to various other homology groups, amongst them Floer homology of the cotangent bundle of the space of gauge equivalence classes of flat $\mathbf{SO}(3)$ connections over Σ .

One difficulty in extending elliptic Morse homology to manifolds Y of arbitrary dimension n consists in the fact that the linearization of equation (4) ceases to be elliptic, even if appropriate gauge fixing conditions are imposed. To overcome this problem we introduce in Section 8 a modification of the above setup. The main idea is to restrict the configuration space $X = \mathcal{A}(P) \times \Omega^{n-2}(Y, \text{ad } P)$ to the Banach submanifold

$$X_1 := \{(A, \omega) \in \mathcal{A}(P) \times \Omega^1(Y, \text{ad}(P)) \mid d_A^* \omega = 0\}$$

of X and consider the flow (4) on X_1 instead. This modification is natural because the critical points of J are automatically contained in X_1 . As it turns out, the linearization obtained through this modification are in fact elliptic equations, however of nonlocal type. A discussion of their compactness and transversality properties is left to future work.

2. ELLIPTIC YANG–MILLS FLOW EQUATIONS

Let Y be a smooth oriented manifold of dimension $n > 1$ without boundary.

For pairs $(A, \omega) \in \mathcal{A}(P) \times \Omega^{n-2}(Y, \text{ad}(P))$ we consider the functional

$$\mathcal{J}(A, \omega) = \int_Y \langle F_A, *\omega \rangle - \frac{1}{2} |\omega|^2 \text{vol}(Y).$$

The L^2 gradient of \mathcal{J} is given by

$$\nabla \mathcal{J}(A, \omega) = ((-1)^{n+1} * d_A \omega, *F_A - \omega).$$

The elliptic Yang–Mills flow equations are

$$(5) \quad \begin{cases} 0 = \partial_s A - d_A \Psi + (-1)^{n+1} * d_A \omega \\ 0 = \partial_s \omega + [\Psi, \omega] - \omega + *F_A \end{cases}$$

for a connection $A \in \mathcal{A}(P)$ and a form $\omega \in \Omega^{n-2}(Y, \text{ad}(P))$.

Remark 1. The factor $(-1)^{n+1}$ appearing in the first equation in (5) results from the different signs the formal adjoint $d_A^* = (-1)^{n(\deg \omega + 1) + 1} * d_A * \omega$ of d_A has in different degrees and dimensions.

Stationary points of this flow equation such that $\Psi = 0$ satisfy

$$(6) \quad d_A \omega = 0 \quad \text{and} \quad \omega = *F_A.$$

Hence in particular the set of critical points of \mathcal{J} is in bijection with the set of Yang–Mills connections on the bundle P .

Remark 2. Assume (A, ω) is a solution of (5) on $I \times Y$, where I is an interval. Then it follows that ω satisfies the second order linear equation

$$(7) \quad 0 = \ddot{\omega} - \dot{\omega} - d_A^* d_A \omega.$$

If in addition $d_A^* \omega = 0$ then this becomes an elliptic equation with the Hodge Laplacian $-\frac{d^2}{ds^2} + \Delta_A$ on $I \times Y$ as leading term. In case $n = 2$ the $d_A^* \omega = 0$ is always satisfied because ω is a 0-form then. We also note that for critical points it always holds that $d_A^* \omega = 0$, because $d_A^* \omega = d_A^* * F_A = 0$ by (6) and the Bianchi identity.

3. MODULI SPACES AND FREDHOLM THEORY

3.1. Gradient flow lines and moduli spaces. Throughout we call any smooth solution (A, ω, Ψ) of (5) on $\mathbb{R} \times \Sigma$ a **negative gradient flow line** of \mathcal{J} . We say that (A, ω, Ψ) is in **temporal gauge** if $\Psi = 0$. The **energy** of a solution (A, ω, Ψ) of (5) is

$$(8) \quad E_f(A, \omega, \Psi) := \frac{1}{2} \int_{\mathbb{R}} \|(-1)^n * d_A \omega + d_A \Psi\|_{L^2(\Sigma)}^2 + \|*F_A - \omega + [\Psi \wedge \omega]\|_{L^2(\Sigma)}^2 ds.$$

In Section 5 it is shown that a solution $(A, \omega, 0)$ of (5) on $\mathbb{R} \times \Sigma$ in temporal gauge has finite energy if and only if there exist critical points (A^\pm, ω^\pm) of \mathcal{J} such that $(A(s), \omega(s))$ converges exponentially to (A^\pm, ω^\pm) as $s \rightarrow \pm\infty$.

We let $\mathcal{M}(A^-, \omega^-, A^+, \omega^+)$ denote the moduli space of gauge equivalence classes of negative gradient flow lines from (A^-, ω^-) to (A^+, ω^+) , i.e.

$$\mathcal{M}(A^-, \omega^-, A^+, \omega^+) := \frac{\widehat{\mathcal{M}}(A^-, \omega^-, A^+, \omega^+)}{\mathcal{G}(P)},$$

where

$$\widehat{\mathcal{M}}(A^-, \omega^-, A^+, \omega^+) := \left\{ (A, \omega) \left| \begin{array}{l} (A, \omega, 0) \text{ satisfies (5), } E_f(A, \omega, 0) < \infty, \\ \lim_{s \rightarrow \pm\infty} (A(s), \omega(s)) \in [(A^\pm, \omega^\pm)] \end{array} \right. \right\}.$$

The moduli space $\mathcal{M}(A^-, \omega^-, A^+, \omega^+)$ arises in a slightly different way as the quotient of all finite energy gradient flow lines (A, ω, Ψ) from (A^-, ω^-) to (A^+, ω^+) such that $\Psi(s) \rightarrow 0$ as $s \rightarrow \pm\infty$ modulo the action of the group $\mathcal{G}(\mathbb{R} \times \Sigma)$ of smooth time-dependent gauge transformations, which converge exponentially to the identity as $s \rightarrow \pm\infty$. Our goal in the subsequent sections is to show that the moduli space $\mathcal{M}(A^-, \omega^-, A^+, \omega^+)$ is a compact manifold and to determine its dimension. For this we first study the linearized operator for Eq. (5).

3.2. Linearized operator. Linearizing the section

$$\mathcal{F}: (A, \omega, \Psi) \mapsto \begin{pmatrix} \partial_s A - d_A \Psi - *d_A \omega \\ \partial_s \omega + [\Psi, \omega] - \omega + *F_A \end{pmatrix}$$

at (A, ω, Ψ) yields the linear operator

$$(9) \quad \mathcal{D}_{(A, \omega, \Psi)}: (\alpha, v, \psi) \mapsto \nabla_s \begin{pmatrix} \alpha \\ v \\ \psi \end{pmatrix} + \underbrace{\begin{pmatrix} *[\omega \wedge \cdot] & -*d_A & -d_A \\ *d_A & -\mathbb{I} & -[\omega \wedge \cdot] \\ -d_A^* & *[\omega \wedge * \cdot] & 0 \end{pmatrix}}_{=: B_{(A, \omega, \Psi)}} \begin{pmatrix} \alpha \\ v \\ \psi \end{pmatrix}$$

where $\nabla_s := \frac{\partial}{\partial s} + [\Psi \wedge \cdot]$. It is straightforward to check that the operator $B_{(A, \omega, \Psi)}$ is symmetric. Below we show that it is a densely defined self-adjoint operator on the Hilbert space L^2 .

Remark 3 (Gauge fixing condition). The last line in (9) is zero precisely if (α, v, ψ) is orthogonal to the gauge orbit through (\mathbb{A}, ω) , where we denote $\mathbb{A} := A + \Psi ds$. This follows from the fact that the action

$$g \cdot (\mathbb{A}, \omega) = (g^* A + (g^{-1} \Psi g + g^{-1} \dot{g}) \wedge ds, g^{-1} \omega g)$$

linearizes as

$$(d_A \varphi + (\dot{\varphi} + [\Psi \wedge \varphi]) \wedge ds, [\omega \wedge \varphi]),$$

where $\varphi \in C^\infty(\mathbb{R}, \Omega^0(M, \text{ad}(P)))$.

We now describe the functional analytic setup in which we consider the operator $\mathcal{D}_{(A, \omega, \Psi)}$. In order to proof the Fredholm theorem below we have to introduce weighted Sobolev spaces. We therefore fix a number $\delta > 0$ and

a smooth cut-off function β such that $\beta(s) = -1$ if $s < 0$ and $\beta(s) = 1$ if $s > 1$. We fix a constant $p > 1$. We denote

$$\mathcal{L}_\delta^p := L_\delta^p(\mathbb{R}, L^p(\Sigma))$$

and

$$\mathcal{W}_\delta^p := W_\delta^{1,p}(\mathbb{R}, L^p(\Sigma)) \cap L_\delta^p(\mathbb{R}, W^{1,p}(\Sigma)),$$

where

$$\begin{aligned} L_\delta^p(\mathbb{R}, L^p(\Sigma)) &:= L_\delta^p(\mathbb{R}, L^p(\Sigma, T^*\Sigma \otimes \text{ad}(P))) \\ &\quad \oplus L_\delta^p(\mathbb{R}, L^p(\Sigma, \text{ad}(P))) \oplus L_\delta^p(\mathbb{R}, L^p(\Sigma, \text{ad}(P))), \end{aligned}$$

and similarly for $W_\delta^{1,p}(\mathbb{R}, L^p(\Sigma))$ and $W_\delta^{1,p}(\mathbb{R}, L^p(\Sigma))$. Multiplication with the function $e^{\delta\beta(s)s}$ yields Banach space isomorphisms

$$\nu_0: \mathcal{L}_\delta^p \rightarrow \mathcal{L}_0^p =: \mathcal{L}^p \quad \text{and} \quad \nu_1: \mathcal{W}_\delta^p \rightarrow \mathcal{W}_0^p =: \mathcal{W}^p.$$

We now define

$$\mathcal{D}_{(A,\omega,\Psi)}^\delta := \nu_1 \circ \mathcal{D}_A \circ \nu_0^{-1}: \mathcal{W}^p \rightarrow \mathcal{L}^p.$$

The operator $\mathcal{D}_{(A,\omega,\Psi)}^\delta$ is Fredholm if and only if this holds for $\mathcal{D}_{(A,\omega,\Psi)}$, in which case both Fredholm indices coincide. Note that the operator $\mathcal{D}_{(A,\omega,\Psi)}^\delta$ takes the form

$$(10) \quad \mathcal{D}_{(A,\omega,\Psi)}^\delta = \frac{d}{ds} + B_{(A,\omega,\Psi)}^\delta(s),$$

where we denote $B_{(A,\omega,\Psi)}^\delta(s) := B_{(A,\omega,\Psi)}(s) - (\beta + \beta's)\delta$. From our assumptions on (A, ω, Ψ) and β it follows that the operator family $s \mapsto B_{(A,\omega,\Psi)}^\delta(s)$ converges to $B_{(A^\pm, \omega^\pm, \Psi^\pm)} \mp \delta$ as $s \rightarrow \pm\infty$. These limit operators are invertible for a suitable choice of $\delta > 0$.

3.3. Fredholm theorem. The aim of this section is to prove the following theorem.

Theorem 4. *Let (A, ω, Ψ) be a solution of (5) and assume that there exist solutions (A^\pm, ω^\pm) of the critical point equation (6) such that*

$$\lim_{s \rightarrow \pm\infty} (A(s), \omega(s)) = (A^\pm, \omega^\pm)$$

in $C^k(\Sigma)$ for every $k \in \mathbb{N}_0$. Then the linear operator $\mathcal{D}_{(A,\omega,\Psi)}$ is a Fredholm operator of index

$$(11) \quad \text{ind } \mathcal{D}_{(A,\omega,\Psi)} = \text{ind } H_{A^-} - \text{ind } H_{A^+}.$$

(Here we denote by $\text{ind } H_{A^\pm}$ the index of the Yang–Mills Hessian H_{A^\pm} of the Yang–Mills connections A^\pm).

Because the statement of this theorem is invariant under gauge transformations it suffices to prove it for $\Psi = 0$. As remarked before Eq. (10), the operator $\mathcal{D}_{(A,\omega,\Psi)}$ can equivalently be replaced by $\mathcal{D}_{(A,\omega,\Psi)}^\delta$. In this situation we denote $B^\delta := B_{(A,\omega,0)}^\delta$ and $\mathcal{D}^\delta := \mathcal{D}_{(A,\omega,0)}^\delta$.

Proof of Theorem 4 in the case $p = 2$. We let $p = 2$. Then $B^\delta(s)$ is a self-adjoint operator on the Hilbert space $H := L^2(\Sigma)$ with domain $\text{dom } B^\delta(s) = W := W^{1,2}(\Sigma)$, for every $s \in \mathbb{R}$. This follows by a straight-forward modification of the proof of Proposition 15.

Proof. [Theorem 4 in the case $p = 2$]. The result follows from [17, Theorem A]. To apply this theorem we need to check that the following properties (i–iv) are satisfied. (i) The inclusion $W \hookrightarrow H$ of Hilbert spaces is compact with dense range. This holds true by definition of the space W and the Rellich–Kontschikov compactness theorem. (ii) The norm of W is equivalent to the graph norm of $B^\delta(s): W \rightarrow H$ for every $s \in \mathbb{R}$. This follows from a standard elliptic estimate for the operator $B^\delta(s): W \rightarrow H$. (iii) The map $\mathbb{R} \rightarrow \mathcal{L}(W, H): s \mapsto B^\delta(s)$ is continuously differentiable with respect to the weak operator topology. For this we need to verify that for every $\xi \in W$ and $\eta \in H$ the map $s \mapsto \langle B^\delta(s)\xi, \eta \rangle$ is of class $C^1(\mathbb{R}, \mathbb{R})$. Because $s \mapsto (A(s), \omega(s))$ is a smooth path in $\mathcal{A}(P) \times \Omega^0(\Sigma, \text{ad}(P))$ (and B^δ depends smoothly on (A, ω)) this property is clearly satisfied. (iv) The operators $B_{(A^\pm, \omega^\pm, 0)}^\delta \in \mathcal{L}(W, H)$ are invertible and are the limits of $B^\delta(s)$ in the norm topology as $s \rightarrow \pm\infty$. Invertibility follows from the choice of the weight δ . The exponential decay Theorem 11 gives uniform convergence $(A(s), \omega(s)) \rightarrow (A^\pm, \omega^\pm)$, hence in particular norm convergence $B^\delta(s) \rightarrow B_{(A^\pm, \omega^\pm, 0)}^\delta$ as $s \rightarrow \pm\infty$. We have thus verified that the operator family $s \mapsto B^\delta(s)$ satisfies all assumptions of [17, Theorem A]. It hence follows that the operator $\mathcal{D}^\delta = \frac{d}{ds} + B^\delta(s)$ is Fredholm with index given by its spectral flow. That this spectral flow is equal to the right-hand side of (19) is the content of the subsequent Lemma 8. \square

It remains to determine the index of the Fredholm operator \mathcal{D}^δ . This index is related to the spectral flow of the operator family $s \mapsto B^\delta(s)$ as we explain next. Here we follow the discussion in [17, Section 4]. Recall that a **crossing** of B^δ is a number $s \in \mathbb{R}$ for which $B^\delta(s)$ is not injective. The **crossing operator** at $s \in \mathbb{R}$ is the map

$$\Gamma(B^\delta, s): \ker B^\delta(s) \rightarrow \ker B^\delta(s), \quad \Gamma(B^\delta, s) = P\dot{B}^\delta(s),$$

where $P: H \rightarrow H$ denotes the orthogonal projection onto $\ker B^\delta(s)$. A crossing $s \in \mathbb{R}$ is called **regular** if $\Gamma(B^\delta, s)$ is nonsingular. It can be shown (cf. [17, Theorem 4.2]) that the operator family $s \mapsto B^\delta(s) + \delta\mathbb{1}$ has only regular crossings for almost every $\delta \in \mathbb{R}$. Hence by choosing the weight $\delta > 0$ appropriately we may assume that the curve B^δ has only regular crossings. Then the number of crossings of B^δ is finite. The **signature** of the crossing $s \in \mathbb{R}$ is the signature (the number of positive minus the number of negative eigenvalues) of the endomorphism $\Gamma(B^\delta, s)$. The Fredholm index of \mathcal{D}^δ is determined by the crossing signatures via the relation

$$(12) \quad \text{ind } \mathcal{D}^\delta = - \sum_s \text{sign } \Gamma(B^\delta, s)$$

where the sum is over all crossings, cf. [17, Theorem 4.1].

We determine the Fredholm index of \mathcal{D}^δ from (12), relating the crossing indices of B^δ to that of a further path of operators $C(s)$ which we introduce next. For $\lambda \in \mathbb{R} \setminus \{-1\}$ and $(A, \omega) = (A(s), \omega(s))$ we define the symmetric operator $C_\lambda(s)$ by

$$C_\lambda(s) := \begin{pmatrix} \frac{1}{\lambda+1} d_A^* d_A + *[\omega \wedge \cdot] & -d_A + \frac{1}{\lambda+1} * d_A[\omega \wedge \cdot] \\ -d_A^* + \frac{1}{\lambda+1} * [\omega \wedge d_A \cdot] & -\frac{1}{\lambda+1} * [\omega \wedge *[\omega \wedge \cdot]] \end{pmatrix}.$$

We furthermore set $C(s) := C_0(s)$.

Lemma 5. *For every $s \in \mathbb{R}$ the crossing indices $\Gamma(B(s))$ and $\Gamma(C(s))$ coincide.*

Proof. We first notice that $\lambda = 0$ is an eigenvalue of $B(s)$ if and only if it is an eigenvalue of $C(s)$. In this case, the corresponding eigenspaces are of the same dimension. These facts follow from Proposition 6 below. It remains to show that for every crossing $s \in \mathbb{R}$ the signatures $\text{sign } \Gamma(B, s)$ and $\text{sign } \Gamma(C, s)$ coincide. We first prove this equality for a simple crossing s_0 , i.e. in the case where the crossing is regular and in addition the kernels of $B(s_0)$ and $C(s_0)$ are one-dimensional. In this situation let $s \mapsto \lambda(s)$, $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$, be a C^1 path of eigenvalues of $B(s)$ with corresponding path of normalized eigenvectors $\xi(s) = (\alpha(s), v(s), \psi(s))$ such that $\lambda(s_0) = 0$. Similarly, let $\mu(s)$ be a C^1 path of eigenvalues with normalized eigenvectors $\tilde{\xi}(s) = (\tilde{\alpha}(s), \tilde{\psi}(s))$ such that $\mu(s_0) = 0$ and $\tilde{\xi}(s_0) = (\alpha(s_0), \psi(s_0))$. It suffices to prove that

$$(13) \quad \text{sign } \dot{\lambda}(s_0) = \text{sign } \dot{\mu}(s_0).$$

To prove this claim we first apply Proposition 6 below which gives the identity

$$(14) \quad C_{\lambda(s)}(s)(\alpha(s), \psi(s)) = \lambda(s)(\alpha(s), \psi(s))$$

for all $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$. Next, by definition of the operator families $C_{\lambda(s)}(s)$ and $C(s)$ it follows that

$$(15) \quad \left. \frac{d}{ds} \right|_{s=s_0} C_{\lambda(s)}(s) = \dot{C}(s_0) + \begin{pmatrix} -\dot{\lambda}(s_0) d_A^* d_A & -\dot{\lambda}(s_0) * d_A[\omega \wedge \cdot] \\ -\dot{\lambda}(s_0) * [\omega \wedge d_A \cdot] & \dot{\lambda}(s_0) * [\omega \wedge *[\omega \wedge \cdot]] \end{pmatrix}.$$

We apply Proposition 7 below to Eq. (14). Together with (15) it then follows that (we abbreviate $A := A(s_0)$, $\omega := \omega(s_0)$ and recall that $\tilde{\xi}(s_0) =$

$$\begin{aligned}
& (\alpha(s_0), \psi(s_0))) \\
\dot{\lambda}(s_0) &= \left\langle \frac{d}{ds} \Big|_{s=s_0} C_{\lambda(s)}(s) \tilde{\xi}(s_0), \tilde{\xi}(s_0) \right\rangle \\
&= \langle \dot{C}(s_0) \tilde{\xi}(s_0), \tilde{\xi}(s_0) \rangle \\
&\quad + \left\langle \begin{pmatrix} -\dot{\lambda}(s_0) d_A^* d_A & -\dot{\lambda}(s_0) * d_A [\omega \wedge \cdot] \\ -\dot{\lambda}(s_0) * [\omega \wedge d_A \cdot] & \dot{\lambda}(s_0) * [\omega \wedge * [\omega \wedge \cdot]] \end{pmatrix} \tilde{\xi}(s_0), \tilde{\xi}(s_0) \right\rangle \\
&= \dot{\mu}(s_0) - \dot{\lambda}(s_0) \langle d_A^* d_A \alpha(s_0), \alpha(s_0) \rangle + 2 \dot{\lambda}(s_0) \langle * d_A \alpha(s_0), [\omega \wedge \psi(s_0)] \rangle \\
&\quad + \dot{\lambda}(s_0) \langle * [\omega \wedge * [\omega \wedge \psi(s_0)]], \psi(s_0) \rangle \\
&= \dot{\mu}(s_0) - \dot{\lambda}(s_0) \|d_A \alpha(s_0)\|^2 - \dot{\lambda}(s_0) \|[\omega \wedge \psi(s_0)]\|^2 \\
&\quad + 2 \dot{\lambda}(s_0) \langle * d_A \alpha(s_0), [\omega \wedge \psi(s_0)] \rangle.
\end{aligned}$$

The term $\dot{\mu}(s_0)$ in the third equation appears as a consequence of Proposition 7 below. For the remaining terms in that line we compute

$$\begin{aligned}
\langle \alpha(s_0), -\dot{\lambda}(s_0) * d_A [\omega \wedge \psi(s_0)] \rangle &= \dot{\lambda}(s_0) \langle * \alpha(s_0), d_A [\omega \wedge \psi(s_0)] \rangle \\
&= \dot{\lambda}(s_0) \langle d_A^* * \alpha(s_0), [\omega \wedge \psi(s_0)] \rangle \\
&= \dot{\lambda}(s_0) \langle * d_A \alpha(s_0), [\omega \wedge \psi(s_0)] \rangle,
\end{aligned}$$

and likewise for $\langle -\dot{\lambda}(s_0) * [\omega \wedge d_A \alpha(s_0)], \psi(s_0) \rangle$. It finally follows that

$$\begin{aligned}
\dot{\mu}(s_0) &= \\
& (1 + \|d_A \alpha(s_0)\|^2 + \|[\omega \wedge \psi(s_0)]\|^2 - 2 \langle * d_A \alpha(s_0), [\omega \wedge \psi(s_0)] \rangle) \dot{\lambda}(s_0),
\end{aligned}$$

where the factor in front of $\dot{\lambda}(s_0)$ is positive (in fact, greater or equal to 1) by the Cauchy-Schwarz inequality. This shows the claimed equality (13) in the case of a simple crossing s_0 . The case of a general regular crossing can be treated similarly. This completes the proof. \square

Proposition 6. *Let $B(s) = B_{(A(s), \omega(s), 0)}$ be the self-adjoint operator as in (9). Then $\xi = (\alpha, v, \psi) \in \text{dom } B(s)$ satisfies the eigenvalue equation $B(s)\xi = \lambda\xi$ for $\lambda \in \mathbb{R} \setminus \{-1\}$ if and only if it satisfies the nonlinear eigenvalue equation*

$$(16) \quad C_\lambda(s)(\alpha, \psi) = \lambda(\alpha, \psi).$$

Proof. Let $\xi = (\alpha, v, \psi) \in \text{dom } B(s)$ satisfy

$$(17) \quad B(s)\xi = \lambda\xi$$

for some $\lambda \neq -1$. Then it follows that

$$(18) \quad v = \frac{1}{\lambda + 1} (* d_A \alpha - [\omega \wedge \psi]).$$

Inserting this v into the first and last of the three equations in (17) yields (16). Conversely, let (α, ψ) solve (16) for some $\lambda \neq -1$. Defining v by

equation (18) and setting $\xi := (\alpha, v, \psi)$ we obtain a solution ξ of (17) for this eigenvalue λ . \square

Proposition 7. *Let $s \mapsto F(s)$, $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$, be a C^1 path of densely defined symmetric operators on some Hilbert space H . Let $s \mapsto \lambda(s)$, $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$, be a C^1 path of eigenvalues of the operator family F with normalized eigenvectors $x(s)$. Then it follows that*

$$\dot{\lambda}(s_0) = \langle \dot{F}(s_0)x(s_0), x(s_0) \rangle.$$

Proof. We differentiate the eigenvalue equation $F(s)x(s) = \lambda(s)x(s)$ to obtain

$$\dot{F}x + F\dot{x} = \dot{\lambda}x + \lambda\dot{x}.$$

Take the inner product of both sides with x and use that by symmetry of F

$$\langle F\dot{x}, x \rangle = \langle \dot{x}, Fx \rangle = \langle \dot{x}, \lambda x \rangle$$

to obtain the result. \square

Lemma 8. *The total number $\sum_{s \in \mathbb{R}} \Gamma(C(s))$ of eigenvalue crossings of the operator family C equals the right-hand side of (19). In particular, the Fredholm index of $\mathcal{D}_{(A, \omega, \Psi)}$ is as stated in (19).*

$$(19) \quad \text{ind } \mathcal{D}_{(A, \omega, \Psi)} = \text{ind } H_{A^-} - \text{ind } H_{A^+}.$$

$$C_\lambda(s) := \begin{pmatrix} \frac{1}{\lambda+1} d_A^* d_A + *[\omega \wedge \cdot] & -d_A + \frac{1}{\lambda+1} * d_A[\omega \wedge \cdot] \\ -d_A^* + \frac{1}{\lambda+1} * [\omega \wedge d_A \cdot] & -\frac{1}{\lambda+1} * [\omega \wedge *[\omega \wedge \cdot]] \end{pmatrix}.$$

Proof. We denote $C^\pm := \lim_{s \rightarrow \pm\infty} C(s)$. By assumption, each pair (A^\pm, ω^\pm) satisfies the critical point equation (6). Hence it follows that

$$C^\pm = \begin{pmatrix} d_{A^\pm}^* d_{A^\pm} + *[*F_{A^\pm} \wedge \cdot] & -d_{A^\pm} + *d_{A^\pm}[*F_{A^\pm} \wedge \cdot] \\ -d_{A^\pm}^* + *[*F_{A^\pm} \wedge d_{A^\pm} \cdot] & - *[*F_{A^\pm} \wedge [F_{A^\pm} \wedge \cdot]] \end{pmatrix}.$$

Note that the upper left entry of C^\pm is equal to the Yang–Mills Hessian H_{A^\pm} . Now it is straightforward to check that the nonzero eigenspaces of C^\pm are spanned by vectors $(\alpha, 0)$ where α is an eigenvector of H_{A^\pm} and vectors $(0, \psi)$ satisfying

$$\begin{pmatrix} -d_{A^\pm} \psi + *d_{A^\pm}[*F_{A^\pm} \wedge \psi] \\ - *[*F_{A^\pm} \wedge [F_{A^\pm} \wedge \psi]] \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ \psi \end{pmatrix}$$

for some $\lambda \neq 0$. Because the endomorphism $\psi \mapsto - *[*F_{A^\pm} \wedge [F_{A^\pm} \wedge \psi]]$ is positive semidefinite, there cannot be any negative eigenspaces of this type. It follows that the number of negative eigenvalues of the operator C^\pm equals that of the Yang–Mills Hessian H_{A^\pm} , i.e. is equal to $\text{ind } H_{A^\pm}$. In particular, the number of negative eigenvalues of C^\pm is finite. This implies that the sum of the crossing signatures of C , $\sum_{s \in \mathbb{R}} \Gamma(C(s))$, is equal to $\text{ind } H_{A^-} - \text{ind } H_{A^+}$. This proves the first claim. The second claim now follows from Lemma 5. \square

Proof of Theorem 4 in the case $1 < p < \infty$.

Proof. The assertion of the theorem in the general case is a consequence of the following standard arguments.

Step 1. *Let N be the vector space*

$$N = \{\xi = (\alpha, v, \psi) \in C^\infty(\mathbb{R} \times \Sigma) \mid \mathcal{D}_{(A, \omega, 0)} \xi = 0, \\ \exists c > 0 \forall s \in \mathbb{R}: \|\xi(s)\|_{L^\infty} + \|\partial_s \xi(s)\|_{L^\infty} + \|\nabla_A \xi(s)\|_{L^\infty} \leq ce^{-\delta|s|}\}.$$

Then $\ker(\mathcal{D}_{(A, \omega, 0)}: \mathcal{W}_\delta^p \rightarrow \mathcal{L}_\delta^p) = N$. In particular, this kernel is finite-dimensional and does not depend on p .

By standard elliptic estimates, any solution of $\mathcal{D}_{(A, \omega, 0)} \xi = 0$ is smooth. Exponential decay in $L^2(\Sigma)$ of such ξ holds by (29). Elliptic bootstrapping arguments then show exponential decay in the form stated. This proves the inclusion $\ker(\mathcal{D}_{(A, \omega, 0)}: \mathcal{W}_\delta^p \rightarrow \mathcal{L}_\delta^p) \subseteq N$. The opposite inclusion is clearly satisfied.

Step 2. *Let N^* be the vector space*

$$N^* = \{\eta = (\beta, w, \omega) \in C^\infty(\mathbb{R} \times \Sigma) \mid \mathcal{D}_{(A, \omega, 0)}^* \eta = 0, \\ \exists c > 0 \forall s \in \mathbb{R}: \|\eta(s)\|_{L^\infty} + \|\partial_s \eta(s)\|_{L^\infty} + \|\nabla_A \eta(s)\|_{L^\infty} \leq ce^{-\delta|s|}\}.$$

Then $\text{coker}(\mathcal{D}_{(A, \omega, 0)}^: \mathcal{W}_\delta^p \rightarrow \mathcal{L}_\delta^p) = N^*$. In particular, this cokernel is finite-dimensional and does not depend on p .*

The statement follows from Step 1 using the reflection $s \mapsto -s$.

Step 3. *The range of the operator $\mathcal{D}_{(A, \omega, 0)}: \mathcal{W}_\delta^p \rightarrow \mathcal{L}_\delta^p$ is closed.*

First, there exists a constant $c = c(A, \omega, p) > 0$ such that for any interval $I = (s_0, s_0 + 1)$ the local elliptic estimate

$$\|\xi\|_{W^{1,p}(I \times \Sigma)} \leq c(\|\mathcal{D}_{(A, \omega, 0)} \xi\|_{L^p(I \times \Sigma)} + \|\xi\|_{L^p(I \times \Sigma)})$$

is satisfied for all $\xi \in W^{1,p}(I \times \Sigma)$. Second, for time-independent $(A, \omega, 0) \equiv (A^\pm, \omega^\pm, 0)$ the operator $\mathcal{D}_{(A, \omega, 0)}^\delta$ is invertible as a bounded operator $W^{1,p}(Z^\pm) \rightarrow L^p(Z^\pm)$, where Z^\pm denotes the half-infinite cylinder $Z^- = (-\infty, 0) \times \Sigma$, respectively $Z^+ = (0, \infty) \times \Sigma$. Both estimates can be combined by a standard cut-off function argument into the estimate

$$\|\xi\|_{\mathcal{W}_\delta^p} \leq c(\|\mathcal{D}_{(A, \omega, 0)} \xi\|_{\mathcal{L}_\delta^p} + \|K\xi\|_{\mathcal{L}_\delta^p}),$$

where $K: \mathcal{W}_\delta^p \rightarrow \mathcal{L}_\delta^p$ is a suitable compact operator. The assertion now follows from the abstract closed range lemma, cf. [18, p. 14].

Step 4. *We prove the theorem.*

By the preceding steps, the operator $\mathcal{D}_{(A, \omega, 0)}: \mathcal{W}_\delta^p \rightarrow \mathcal{L}_\delta^p$ is a Fredholm operator, its index being independent of p . Hence the asserted index formula (19) follows from the case $p = 2$. \square

4. COMPACTNESS

In this section we prove a compactness theorem for solutions of (5) with uniformly bounded energy \mathcal{J} . Let $I \subseteq \mathbb{R}$ be an interval. For convenience we shall identify at several instances the path $(A, \Psi) \in C^\infty(I, \mathcal{A}(P) \times \Omega^0(\Sigma, \text{ad}(P)))$ with the connection $\mathbb{A} := A + \Psi ds \in \mathcal{A}(I \times P)$. Its curvature is

$$(20) \quad F_{\mathbb{A}} = F_A + \dot{A} \wedge ds + d_A \Psi \wedge ds \in \Omega^2(I \times \Sigma, \text{ad}(I \times P)).$$

We call a connection \mathbb{A}_1 to be in *local slice with respect to the reference connection* \mathbb{A} if the difference $\mathbb{A}_1 - \mathbb{A} = \alpha + \psi ds$ satisfies

$$d_{\mathbb{A}}^*(\alpha + \psi ds) = 0.$$

This condition is equivalent to

$$(21) \quad \nabla_s \psi - d_A^* \alpha = 0,$$

where we denote $\nabla_s \psi := \partial_s \psi + [\Psi, \psi]$.

In the following we fix $(A_0, \Psi_0, \omega_0) \in \mathcal{A}(P) \times \Omega^0(\Sigma, \text{ad}(P)) \times \Omega^0(\Sigma, \text{ad}(P))$ as a smooth time-independent reference point and denote $\mathbb{A}_0 := A_0 + \Psi_0 ds$. Let $(A, \Psi, \omega) = (A_0, \Psi_0, \omega_0) + (\alpha, \psi, v)$ be a smooth solution (5) on $I \times \Sigma$. We augment (5) with the local slice condition (21) with respect to the reference connection \mathbb{A}_0 . Expanding this system of equations about (A_0, Ψ_0, ω_0) we obtain

$$(22) \quad 0 = \mathcal{F}(A_0, \Psi_0, \omega_0) + (\nabla_s + B_{(A_0, \omega_0, \Psi_0)}) \begin{pmatrix} \alpha \\ v \\ \psi \end{pmatrix} + Q_{\omega_0} \begin{pmatrix} \alpha \\ v \\ \psi \end{pmatrix},$$

where the linear operator $B_{(A_0, \omega_0, \Psi_0)}$ is as defined in (9) and where we define

$$\mathcal{F}(A_0, \Psi_0, \omega_0) := \begin{pmatrix} \dot{A}_0 - d_{A_0} \Psi_0 - *d_{A_0} \omega_0 \\ \dot{\omega}_0 + [\Psi_0, \omega_0] - \omega_0 + *F_{A_0} \\ 0 \end{pmatrix}$$

and

$$Q_{\omega_0} \begin{pmatrix} \alpha \\ v \\ \psi \end{pmatrix} := \begin{pmatrix} -[\alpha \wedge \psi] - *[\alpha \wedge v] \\ [\psi, v] + \frac{1}{2} *[\alpha \wedge \alpha] \\ -*[\omega_0 \wedge *v] \end{pmatrix}.$$

We define the *gauge-invariant energy density* of the solution (A, Ψ, ω) of (5) to be

$$(23) \quad e(A, \Psi, \omega) := \frac{1}{2}(|d_A \omega - *d_A \Psi|^2 + |*F_A - \omega + [\Psi, \omega]|^2): I \times \Sigma \rightarrow \mathbb{R}.$$

Proposition 9. *Let $I \subseteq \mathbb{R}$ be a compact interval and $(A^\nu, 0, \omega^\nu)$ be a sequence of smooth solutions of Eq. (5) on $I \times \Sigma$ in temporal gauge. Assume that there exists a constant $C > 0$ such that for $e^\nu := e(A^\nu, 0, \omega^\nu)$*

$$(24) \quad \|e^\nu\|_{L^\infty(I \times \Sigma)} \leq C$$

for all $\nu \in \mathbb{N}$. Then there exists a subsequence, still denoted by $(A^\nu, 0, \omega^\nu)$, and a sequence of gauge transformations $g^\nu \in \mathcal{G}(I \times P)$ such that the sequence $(g^\nu)^(A^\nu, 0, \omega^\nu)$ converges in the C^k topology, for every $k \in \mathbb{N}_0$.*

Proof. By (20) the curvature of the connection $\mathbb{A}^\nu := A^\nu + 0 ds \in \mathcal{A}(I \times P)$ is

$$F_{\mathbb{A}^\nu} = F_{A^\nu} + *d_{A^\nu}\omega^\nu \wedge ds.$$

Assumption (24) together with the definition (23) of e^ν implies the uniform curvature bound

$$\begin{aligned} \|F_{\mathbb{A}^\nu}\|_{L^\infty(I \times \Sigma)} &\leq \\ \|F_{A^\nu} - \omega^\nu\|_{L^\infty(I \times \Sigma)} + \|\omega^\nu\|_{L^\infty(I \times \Sigma)} + \|d_{A^\nu}\omega^\nu\|_{L^\infty(I \times \Sigma)} &\leq C_1. \end{aligned}$$

for some further constant C_1 . To estimate the term $\|\omega^\nu\|_{L^\infty(I \times \Sigma)}$ we used (51). Therefore the assumptions of Uhlenbeck's weak compactness theorem (cf. [29, Theorem A]) are satisfied, for any Sobolev exponent $p > \frac{3}{2}$. It yields the existence of a sequence of gauge transformations $g^\nu \in \mathcal{G}^{2,p}(I \times P)$ such that after passing to a subsequence

$$(g^\nu)^*\mathbb{A}^\nu \rightharpoonup \mathbb{A}^* = A^* + \Psi^* ds \quad (\nu \rightarrow \infty)$$

weakly in $W^{1,p}(I \times \Sigma)$ for some limiting connection $\mathbb{A}^* \in W^{1,p}(I \times \Sigma)$. Let $g_0 \in \mathcal{G}^{2,p}(I \times \Sigma)$ be a gauge transformation such that $\tilde{\mathbb{A}}_0 := g_0^*\mathbb{A}^*$ is in temporal gauge, i.e. of the form $\tilde{\mathbb{A}}_0 = \tilde{A}_0 + 0 ds$. Let $\mathbb{A}_0 = A_0 + 0 ds$ be a smooth connection $W^{1,p}$ close to $\tilde{\mathbb{A}}_0$. Because the weakly convergent sequence $(g^\nu)^*\mathbb{A}^\nu$ is bounded in $W^{1,p}$, this is also the case for the gauge transformed sequence $(g^\nu g_0)^*\mathbb{A}^\nu$. In the following we rename $(g^\nu g_0)^*\mathbb{A}^\nu$ to A^ν , and likewise $(g^\nu g_0)^{-1}\omega^\nu g^\nu g_0$ to ω^ν . Now the local slice theorem, cf. [29, Theorem F], yields a further sequence h^ν of gauge transformations such that $(g^\nu g_0 h^\nu)^*\mathbb{A}^\nu$ is a bounded sequence in $W^{1,p}$, which in addition is in local slice with respect to the reference connection \mathbb{A}_0 . Denoting $\alpha^\nu + \psi^\nu ds := (g^\nu g_0 h^\nu)^*\mathbb{A}^\nu - \mathbb{A}_0$ this means that

$$(25) \quad d_{\mathbb{A}_0}^*(\alpha^\nu + \psi^\nu ds) = d_{A_0}^*\alpha^\nu - \dot{\psi}^\nu = 0$$

for all $\nu \in \mathbb{N}$. We furthermore choose $\omega_0 \in \Omega^0(\Sigma, \text{ad}(P))$ as a smooth reference point and set $v^\nu := \omega^\nu - \omega_0$. Using again the uniform estimate (51) and assumption (24) it follows that the sequence v^ν is uniformly bounded in $W^{1,p}(I \times \Sigma)$. From (22) and (25) it follows that each $(\alpha^\nu, v^\nu, \psi^\nu)$ satisfies

the equation

$$(26) \quad (\nabla_s + B_{(A_0, \omega_0, 0)}) \begin{pmatrix} \alpha^\nu \\ v^\nu \\ \psi^\nu \end{pmatrix} = -\mathcal{F}(A_0, 0, \omega_0) - Q_{\omega_0} \begin{pmatrix} \alpha^\nu \\ v^\nu \\ \psi^\nu \end{pmatrix}.$$

The right-hand side of (26) is uniformly bounded in $W^{1,p}(I \times \Sigma)$ since $\mathcal{F}(A_0, 0, \omega_0)$ is smooth, $Q_{\omega_0}: W^{1,p}(I \times \Sigma) \rightarrow W^{1,p}(I \times \Sigma)$ is bounded (for p sufficiently large) and $(\alpha^\nu, v^\nu, \psi^\nu)$ satisfies a uniform bound in $W^{1,p}(I \times \Sigma)$. The claim now follows by ellipticity of the linear operator $\nabla_s + B_{(A_0, \omega_0, 0)}$ together with a bootstrap argument. \square

The following theorem states compactness of moduli spaces up to convergence to broken trajectories.

Theorem 10. *Let f be a perturbation such that every critical point of $\mathcal{J} + h_f$ is nondegenerate. Let $(A^\nu, 0, \omega^\nu)$ be a sequence in $\widetilde{\mathcal{M}}(A_-^\nu, A_+^\nu)$ with bounded energy*

$$(27) \quad \sup_\nu (\mathcal{J}(A^\nu, \omega^\nu) + h_f(A^\nu, \omega^\nu)) < \infty.$$

Assume that the two sequences of critical points A_\pm^ν converge uniformly to critical points $A^\pm \in \text{Crit}(\mathcal{J} + h_f)$. Then there is a subsequence, still denoted by $(A^\nu, 0, \omega^\nu)$, critical points $A^- = B_0, \dots, B_\ell = A^+ \in \text{Crit}(\mathcal{J} + h_f)$ and connecting trajectories $(A_i, \Psi_i, \omega_i) \in \widetilde{\mathcal{M}}(B_i, B_{i+1})$ for $i = 0, \dots, \ell - 1$, such that $(A^\nu, 0, \omega^\nu)$ converges to the broken trajectory

$$((A_0, \Psi_0, \omega_0), \dots, (A_{\ell-1}, \Psi_{\ell-1}, \omega_{\ell-1}))$$

in the following sense.

For every $i = 0, \dots, \ell - 1$ there is a sequences $s_i^\nu \in \mathbb{R}$ and a sequence of gauge transformations $g_i^\nu \in \mathcal{G}(\mathbb{R} \times \Sigma)$ such that the sequence $(g_i^\nu)^(A^\nu(\cdot + s_i^\nu), 0, \omega^\nu(\cdot + s_i^\nu))$ converges to (A_i, Ψ_i, ω_i) uniformly on compact subsets of $\mathbb{R} \times \Sigma$.*

Proof. Assumption (27) implies the uniform bound

$$\|e_I^\nu\|_{L^\infty(I \times \Sigma)} \leq C(I)$$

for the sequence of energy densities $e_I^\nu := e(A^\nu|_I, \omega^\nu|_I)$, for every compact interval $I \subseteq \mathbb{R}$. This follows because each e_I^ν satisfies the differential inequality (52) on $I \times \Sigma$. Hence the elliptic mean value inequality, cf. Theorem 20, applies and yields a pointwise bound for e_I^ν in terms of $\|e_I^\nu\|_{L^2(I \times \Sigma)}^2$. The latter quantity is uniformly bounded in ν by (27) and the identity

$$\mathcal{J}(A_-^\nu, \omega_-^\nu) - \mathcal{J}(A_+^\nu, \omega_+^\nu) = \int_{-\infty}^{\infty} \|e^\nu(s)\|_{L^2(\Sigma)}^2 ds.$$

Hence the restriction of (A^ν, ω^ν) to each compact interval $I \subseteq \mathbb{R}$ satisfies the assumption of Proposition 9. Therefore, after passing to a subsequence and modification by gauge transformations it follows that $(A^\nu|_I, \omega^\nu|_I)$ converges uniformly to some smooth $(A^*|_I, \Psi^*|_I, \omega^*|_I)$, which again is a solution of (5)

on $I \times \Sigma$. The remaining parts of the statement now follow from standard arguments, involving the Exponential Decay Theorem 11. \square

5. EXPONENTIAL DECAY

Theorem 11. *We assume that every critical point of \mathcal{J} is nondegenerate and fix a regular value a of \mathcal{J} . Then there exist positive constants δ , $\tilde{\delta}$, and c such that the following holds. For any negative gradient flow line (A, ω, Ψ) with $\mathcal{J}(A(s), \omega(s)) < a$ which satisfies*

$$(28) \quad \|\partial_s A(s) - d_{A(s)} \Psi(s)\|_{L^\infty(Y)} + \|\nabla_s \omega(s)\|_{L^\infty(Y)} \leq \tilde{\delta}$$

for $|s| > s_0$ for a given $s_0 > 0$, then for every C^2 $\xi = (\alpha, v, \psi)$, with $\alpha(s) \in \Omega^1(Y, \text{ad}(P))$, $v(s), \psi(s) \in \Omega^0(Y, \text{ad}(P))$, which satisfies $\mathcal{D}_{(A, \omega, \Psi)}(\xi) = 0$ and does not diverge as $s \rightarrow \pm\infty$,

$$(29) \quad \|\xi\|_{L^2(Y)} \leq ce^{-\delta|s|}$$

for all $s > s_0$.

Proof. The idea is to show that

$$(30) \quad f(s) := \frac{1}{2} \int_Y (|\alpha|^2 + |v|^2 + |\psi|^2) \, \text{dvol}_Y$$

satisfies

$$(31) \quad f''(s) \geq \rho^2 f(s)$$

for $s \geq 1$. Then, this implies that f has exponential decay, because, since

$$\partial_s (e^{-\rho s} (f'(s) + \rho f(s))) = e^{-\rho s} (-\rho^2 f(s) + f''(s)) \geq 0,$$

$f'(s) + \rho f(s) < 0$ (otherwise $e^{-\rho s} (f'(s) + \rho f(s))$ would be positive and increase; thus, since $f(s)$ is bounded, $e^{-\rho s} f(s)$ would decrease and hence $f'(s)$ would increase. Therefore $f(s)$ would be unbounded which is a contradiction and hence $e^{\rho s} f(s)$ is decreasing. Therefore, if the function f satisfies (31), then

$$(32) \quad f(s) \leq e^{-\rho(s-1)} c_1$$

with $c_1 = f(1)$. First, before proving (29), we remark that the assumption (28) and the negative gradient flow equation (5) imply that

$$(33) \quad \|d_A \omega\|_{L^\infty(Y)} + \|\ast F_A - \omega\|_{L^\infty(Y)} \leq \tilde{\delta}$$

and therefore the conditions of the Lemma 12 are satisfied. Thus, if we derive f twice with respect to s , we obtain

$$\begin{aligned}
 f''(s) &= \int_Y (|\nabla_s \alpha|^2 + |\nabla_s v|^2 + |\nabla_s \psi|^2) \, \text{dvol}_Y \\
 &\quad + \int_Y (\langle \alpha, \nabla_s \nabla_s \alpha \rangle + \langle v, \nabla_s \nabla_s v \rangle + \langle \psi, \nabla_s \nabla_s \psi \rangle) \, \text{dvol}_Y \\
 &= 2 \|d_A^* \alpha - *[\omega \wedge *v]\|^2 + 2 \|*d_A v + d_A \psi - *[\omega \wedge \alpha]\|^2 \\
 &\quad + 2 \|*d_A \alpha - v - [\omega \wedge \psi]\|^2 + \langle \alpha, *[(\partial_s A - d_A \Psi) \wedge v] \rangle \\
 (34) \quad &\quad + \langle \alpha, [(\partial_s A - d_A \Psi) \wedge \psi] \rangle - \langle \alpha, *[\nabla_s \omega \wedge \alpha] \rangle \\
 &\quad + \langle v, [\nabla_s \omega \wedge \psi] \rangle - \langle \psi, [(\partial_s A - d_A \Psi) \wedge \alpha] \rangle - \langle \psi, *[\nabla_s \omega \wedge v] \rangle \\
 &\geq 2 \|d_A^* \alpha - *[\omega \wedge *v]\|^2 + 2 \|*d_A v + d_A \psi - *[\omega \wedge \alpha]\|^2 \\
 &\quad + 2 \|*d_A \alpha - v - [\omega \wedge \psi]\|^2 \\
 &\quad - c\tilde{\delta} (\|\alpha\|^2 + \|v\|^2 + \|\psi\|^2)
 \end{aligned}$$

where the second equality follows from the condition $\mathcal{D}_{(A, \omega, \Psi)}(\xi) = 0$ together with the computation (35) below. The final inequality follows from a short computation, using the assumption (28). In fact applying the flow equation (5) several times we obtain

$$\begin{aligned}
 &\langle \alpha, \nabla_s \nabla_s \alpha \rangle + \langle v, \nabla_s \nabla_s v \rangle + \langle \psi, \nabla_s \nabla_s \psi \rangle \\
 &= - \langle \alpha, \nabla_s (*[\omega \wedge \alpha] - *d_A v - d_A \psi) \rangle \\
 &\quad + \langle v, \nabla_s (*d_A \alpha - v - [\omega \wedge \psi]) \rangle - \langle \psi, \nabla_s (d_A^* \alpha - *[\omega \wedge *v]) \rangle \\
 &= \langle \alpha, *d_A \nabla_s v + [\nabla_s, *d_A]v \rangle + \langle \alpha, d_A \nabla_s \psi + [\nabla_s, d_A]\psi \rangle \\
 &\quad - \langle \alpha, *[\nabla_s \omega \wedge \alpha] + *[\omega \wedge \nabla_s \alpha] \rangle - \langle v, *d_A \nabla_s \alpha + [\nabla_s, *d_A]\alpha \rangle \\
 &\quad - \langle v, *d_A \alpha - v - [\omega \wedge \psi] \rangle + \langle v, [\nabla_s \omega \wedge \psi] + [\omega \wedge \nabla_s \psi] \rangle \\
 &\quad + \langle \psi, d_A^* \nabla_s \alpha + [\nabla_s, d_A^*]\alpha \rangle - \langle \psi, *[\nabla_s \omega \wedge *v] + *[\omega \wedge * \nabla_s v] \rangle \\
 &= \langle *d_A \alpha, *d_A \alpha - v - [\omega \wedge \psi] \rangle + \langle \alpha, *[(\partial_s A - d_A \Psi) \wedge v] \rangle \\
 &\quad + \langle d_A^* \alpha, d_A^* \alpha - *[\omega \wedge *v] \rangle + \langle \alpha, [(\partial_s A - d_A \Psi) \wedge \psi] \rangle \\
 &\quad - \langle \alpha, *[\nabla_s \omega \wedge \alpha] \rangle + \langle \alpha, *[\omega \wedge (*[\omega \wedge \alpha] - *d_A v - d_A \psi)] \rangle \\
 &\quad - \langle v, *d_A \nabla_s \alpha \rangle - \langle v, [(\partial_s A - d_A \Psi) \wedge \alpha] \rangle \\
 &\quad - \langle v, *d_A \alpha - v - [\omega \wedge \psi] \rangle \\
 &\quad + \langle v, [\nabla_s \omega \wedge \psi] + [\omega \wedge (d_A^* \alpha - *[\omega \wedge *v])] \rangle \\
 &\quad + \langle d_A \psi, -(*[\omega, \alpha] - *d_A v - d_A \psi) \rangle + \langle \psi, -[(\partial_s A - d_A \Psi) \wedge \alpha] \rangle \\
 &\quad - \langle \psi, *[\nabla_s \omega \wedge *v] - *[\omega \wedge (*d_A \alpha - v - [\omega \wedge \psi])] \rangle
 \end{aligned}$$

$$\begin{aligned}
(35) \quad &= \|d_A^* \alpha - *[\omega \wedge *v]\|^2 + \|*d_A v + d_A \psi - *[\omega \wedge \alpha]\|^2 \\
&+ \|*d_A \alpha - v - [\omega \wedge \psi]\|^2 + \langle \alpha, *[(\partial_s A - d_A \Psi) \wedge v] \rangle \\
&+ \langle \alpha, [(\partial_s A - d_A \Psi) \wedge \psi] \rangle - \langle \alpha, *[\nabla_s \omega \wedge \alpha] \rangle \\
&+ \langle v, [\nabla_s \omega \wedge \psi] \rangle - \langle \psi, [(\partial_s A - d_A \Psi) \wedge \alpha] \rangle - \langle \psi, *[\nabla_s \omega \wedge v] \rangle
\end{aligned}$$

Thus, by Lemma 12 and upon choosing $\tilde{\delta}$ and δ small enough

$$(36) \quad f''(s) \geq \delta^2 (\|\alpha\|^2 + \|v\|^2 + \|\psi\|^2) = \delta^2 f(s),$$

and thus the claimed exponential convergence follows. \square

Lemma 12. *We assume that every critical point of \mathcal{J} is nondegenerate and fix a regular value a of \mathcal{J} . Then there are positive constants $\tilde{\delta}$ and c such that the following holds. For all $(A, \omega) \in \mathcal{A}(P) \times \Omega^{n-2}(Y, \text{ad}(P))$ with $\mathcal{J}(A, \omega) < a$ that satisfy*

$$(37) \quad \|d_A \omega\|_{L^\infty(Y)} + \|*F_A - \omega\|_{L^\infty(Y)} \leq \tilde{\delta}$$

and for every C^2 $\xi = (\alpha, v, \psi)$, with $\alpha \in \Omega^1(Y, \text{ad}(P))$, $v, \psi \in \Omega^0(Y, \text{ad}(P))$, which does not diverge as $s \rightarrow \pm\infty$,

$$\begin{aligned}
(38) \quad &\|\alpha\|_{L^2(Y)}^2 + \|v\|_{L^2(Y)}^2 + \|\psi\|_{L^2(Y)}^2 \\
&\leq c \left(\|d_A^* \alpha - *[\omega \wedge *v]\|_{L^2(Y)}^2 + \|*d_A v + d_A \psi - *[\omega \wedge \alpha]\|_{L^2(Y)}^2 \right. \\
&\quad \left. + \|*d_A \alpha - v - [\omega \wedge \psi]\|_{L^2(Y)}^2 \right).
\end{aligned}$$

Proof. We assume by contradiction that the lemma does not hold. Then there are two sequences $(A_\nu, \omega_\nu) \in \mathcal{A}(P) \times \Omega^{n-2}(Y, \text{ad}(P))$ and $\delta_\nu \rightarrow 0$ such that

$$(39) \quad \|d_{A_\nu} \omega_\nu\|_{L^\infty(Y)} + \|*F_{A_\nu} - \omega_\nu\|_{L^\infty(Y)} \leq \delta_\nu.$$

and (38) does not hold for $c_\nu = \nu$. Since

$$\begin{aligned}
(40) \quad &\frac{1}{2} \|\omega\|_{L^2(Y)}^2 = \mathcal{J}(A, \omega) - \int \langle * \omega \wedge (F_A - * \omega) \rangle \\
&\leq a + \frac{1}{4} \|\omega\|_{L^2(Y)}^2 + \|* \omega - F_A\|_{L^2(Y)}^2,
\end{aligned}$$

by (37), ω_ν and F_{A_ν} are bounded in the L^2 -norm. Because $d_A * \omega_\nu = 0$, ω_ν is bounded in $W^{1,2}$, too. Furthermore, by (37) F_{A_ν} is also bounded in $W^{1,2}$ and thus by Uhlenbeck's weak compactness theorem (cf. [29, Theorem A]) we can assume that A_ν has a weakly convergent subsequence in $\mathcal{A}^{1,p}(P)$ for any given $2 < p < \infty$. Therefore, there is a subsequence of (A_ν, ω_ν) which converges in $W^{1,p} \times W^{1,2}$ to a critical point of \mathcal{J} for which (38) holds for a given c , because the critical points are not degenerate. Thus, since the subsequence converges also in L^∞ there is a ν_0 such that (38) holds for any (A_ν, ω_ν) with $\nu > \nu_0$, which is a contradiction. \square

6. ELLIPTIC YANG–MILLS HOMOLOGY

We fix a compact oriented Riemannian surface (Σ, g) , a compact Lie group G , a principal G -bundle $P \rightarrow \Sigma$, and a regular value $a > 0$ of \mathcal{J} . Associated to these data we define the elliptic Yang–Mills homology $HYM_*(\Sigma, g, P, a)$ as follows. Let

$$\mathcal{R}^a := \frac{\{(A, \omega) \in \mathcal{A}(P) \times \Omega^{n-2}(\Sigma, \text{ad}(P)) \mid d_A^* F_A = 0, \omega = *F_A, \mathcal{J}(A, \omega) < a\}}{\mathcal{G}(P)}$$

denote the (finite) set of gauge equivalence classes of critical points of \mathcal{J} of energy less than a . We assume that the elements of \mathcal{R}^a are isolated in $(\mathcal{A}(P) \times \Omega^{n-2}(\Sigma, \text{ad}(P)))/\mathcal{G}(P)$. They then generate a chain complex

$$CYM_*(\Sigma, g, P, a) := \sum_{[(A, \omega)] \in \mathcal{R}^a} \mathbb{Z}_2 \langle [(A, \omega)] \rangle$$

with grading given by the index of the Yang–Mills Hessian $H_A \mathcal{Y}\mathcal{M}$. For a pair $(A^-, \omega^-, A^+, \omega^+) \in \mathcal{R}^a$ of critical points we let

$$\mathcal{M}(A^-, \omega^-, A^+, \omega^+) = \widetilde{\mathcal{M}}(A^-, \omega^-, A^+, \omega^+)/\mathcal{G}(P)$$

be the moduli space as defined in Section 3. We assume it here to be a smooth manifold which is then of dimension $\text{ind } A^- - \text{ind } A^+$. The group \mathbb{R} acts freely on the moduli space by time-shifts. If this index difference equals 1 it follows by compactness (cf. Theorem 10) that $\mathcal{M}(A^-, \omega^-, A^+, \omega^+)$ modulo the \mathbb{R} -action consists of a finite number of points. For $k \in \mathbb{N}$ we hence obtain a well-defined boundary operator

$$\partial_k : CYM_k(\Sigma, g, P, a) \rightarrow CYM_{k-1}(\Sigma, g, P, a)$$

to be the linear extension of the map

$$\partial_k x := \sum_{\substack{x' \in \mathcal{R}^a \\ \text{ind}(x') = k-1}} n(x, x') x',$$

with $x \in \mathcal{R}^a$ is a critical point of index $\text{ind}(x) = k$. Here $n(x, x') \in \mathbb{Z}_2$ is defined to be the number of elements in $\mathcal{M}(A^-, \omega^-, A^+, \omega^+)$, counted modulo 2, i.e.

$$n(x, x') := \#\mathcal{M}(A^-, \omega^-, A^+, \omega^+) \pmod{2}.$$

Lemma 13. *The map ∂_* satisfies $\partial_* \circ \partial_{*+1} = 0$, i.e. is a chain map.*

Proof. The chain map property follows from standard arguments making use of Theorem 11 on exponential decay of finite energy gradient flow lines. \square

We define the **elliptic Yang–Mills homology associated with** (Σ, g, a, f) to be the collection of abelian groups

$$HYM_k(\Sigma, g, P, a) := \frac{\ker \partial_k}{\text{im } \partial_{k+1}}$$

for $k \in \mathbb{N}_0$.

We conclude this section with a number of remarks.

- We expect standard cobordism arguments to show independence of the homology groups $HYM_*(\Sigma, g, P, a)$ on the choice of the Riemannian metric g and perturbations used to define it.
- To keep the exposition short we developed here a version of elliptic Yang–Mills homology using coefficients in \mathbb{Z}_2 . Defining a variant of it with coefficients in \mathbb{Z} requires to deal with oriented moduli spaces.
- It would be very interesting to compare elliptic Yang–Mills homology with the Morse homology obtained from the Yang–Mills gradient flow on a Riemannian surface and first considered by Atiyah and Bott [3]. The chain complexes in both cases are generated by Yang–Mills connections A (respectively pairs (A, ω) where A is Yang–Mills and $\omega = *F_A$) and hence coincide. However the boundary operators are very different, involving a parabolic equation in the classical situation, in contrast to the elliptic system in the approach presented here.

7. THREE-DIMENSIONAL PRODUCT MANIFOLDS

So far, we defined elliptic Yang–Mills homology in the case $n = 2$ which allows the equations to be nicely elliptic. In the subsequent two sections we discuss two approaches for the case $n = 3$; first we explain how the homology can be applied to a special case of three-dimensional products $Y = \Sigma \times S^1$. Then in Section 8 we show how to restore ellipticity by restricting the space $X = \mathcal{A}(P) \times \Omega^{n-2}(Y, \text{ad}(P))$ to a certain Banach submanifold.

For the first purpose we choose the 3-manifold $\Sigma \times S^1$ with the partially rescaled metric $\varepsilon^2 g_\Sigma \oplus g_{S^1}$, for a parameter $\varepsilon > 0$, and consider the principal $\mathbf{SO}(3)$ -bundle $P \times S^1 \rightarrow \Sigma \times S^1$. The motivation for considering this setup comes from the fact that the perturbed Yang–Mills connections of P and the resulting elliptic homology groups are strongly related to the perturbed geodesics and the homology of the loop group $\mathcal{LM}^g(P)$ of the moduli space of flat connections $\mathcal{M}^g(P)$.

Remark 14. The moduli space $\mathcal{M}^g(P)$ for a principal bundle P over a Riemann surface Σ of genus g was investigated for the first time in 1983 by Atiyah and Bott (cf. [3]). They noticed that the conformal structure of Σ gives rise to a Riemannian metric and an almost complex structure on it; moreover, if a nontrivial principal $\mathbf{SO}(3)$ -bundle P is chosen, then the moduli space $\mathcal{M}^g(P)$, defined as the quotient of the space of the flat connections $\mathcal{A}_0(P) \subseteq \mathcal{A}(P)$ and the identity component $\mathcal{G}_0(P)$ of the group of gauge transformations, is a smooth compact Kähler manifold of dimension $6g - 6$ (cf. [7]). In the nineties some aspects of the topology of $\mathcal{M}^g(P)$ were investigated by Dostoglou and Salamon (cf. [7, 8, 9, 19]), who proved among other results the Atiyah–Floer conjecture, as well as by Hong (cf. [11]).

In fact, on the one hand, there is a bijection between the perturbed Yang–Mills connections on the bundle $P \times S^1$ and the perturbed geodesics on $\mathcal{M}^g(P)$ (cf. [14]). On the other hand it turned out that the Morse homologies, defined using the perturbed L^2 Yang–Mills flow on $P \times S^1 \rightarrow \Sigma \times S^1$ and the heat flow on $\mathcal{M}^g(P)$, are isomorphic (cf. [12, 13]), provided that ε is small enough and that an energy bound b is chosen, i.e.:

$$(41) \quad HM_* \left(\mathcal{L}^b \mathcal{M}^g(P), \mathbb{Z}_2 \right) \cong HM_* \left(\mathcal{A}^{\varepsilon, b} (P \times S^1) / \mathcal{G}_0 (P \times S^1), \mathbb{Z}_2 \right)$$

where $\mathcal{L}^b \mathcal{M}^g(P) \subseteq \mathcal{L} \mathcal{M}^g(P)$ and $\mathcal{A}^{\varepsilon, b} (P \times S^1) \subseteq \mathcal{A} (P \times S^1)$, respectively, denote the subsets with energies bounded from above by b .

Furthermore, in the case of loop spaces the homology is well-defined due to the works of Salamon and Weber (cf. [20, 31]). On the other hand, for the Yang–Mills case, the flow exists if the base manifold is two- or three-dimensional or if it has a symmetry of codimension three (cf. [15, 16]), but Morse–Smale transversality is not proven yet and thus the homology $HM_* (\mathcal{A}^{\varepsilon, b} (P \times S^1) / \mathcal{G}_0 (P \times S^1))$ might not be defined in the general case. In the chosen special case, however, the unstable manifolds of gauge equivalence classes in $\mathcal{A}^{\varepsilon, b} (P \times S^1) / \mathcal{G}_0 (P \times S^1)$ inherit the orientation and the transversality properties from the unstable manifolds of $\mathcal{L}^b \mathcal{M}^g(P)$.

By works of Viterbo (cf. [28]), Salamon and Weber (cf. [20]) and Abbondandolo and Schwarz (cf. [1, 2]) (Morse) homology of $\mathcal{L}^b \mathcal{M}^g(P)$ is isomorphic to Floer homology of the cotangent bundle $T^* \mathcal{M}^g(P)$ defined via the Hamiltonian H_V given by the sum of kinetic and potential energy and considering only orbits with action bounded by b . Moreover, Weber (cf. [31]) showed that the Morse homology of the loop space defined by the heat flow is isomorphic to its singular homology. Summarizing we therefore have (conjecturally) the following isomorphisms of abelian groups:

$$(42) \quad \begin{array}{ccc} H_* (\mathcal{L}^b \mathcal{M}^g(P)) & & H_* (\mathcal{A}^{\varepsilon, b} / \mathcal{G}_0) \\ \cong & & \stackrel{?}{\cong} \\ HM_* (\mathcal{L}^b \mathcal{M}^g(P)) & \cong & HM_* (\mathcal{A}^{\varepsilon, b} / \mathcal{G}_0) \\ \cong & & \\ HF_*^b (T^* \mathcal{M}^g(P), H_V) & & \end{array}$$

Here we denoted $\mathcal{A}^{\varepsilon, b} (P \times S^1) / \mathcal{G}_0 (P \times S^1)$ by $\mathcal{A}^{\varepsilon, b} / \mathcal{G}_0$. It is still an open question whether the Morse homology of $\mathcal{A}^{\varepsilon, b} / \mathcal{G}_0$ defined using the L^2 flow of the Yang–Mills functional is isomorphic to singular homology. In this context, we may introduce the Morse homology $HYM(\Sigma \times S^1, \varepsilon^2 g_\Sigma \oplus g_{S^1}, a, f)$ defined from the functional \mathcal{J} on $\mathcal{A}(P \times S^1) \times \Omega^1(\Sigma \times S^1, \text{ad}(P))$, which can be seen as the Yang–Mills analogue of the cotangent bundle. In fact, in this special case it might be possible to show that $HYM(\Sigma \times S^1, \varepsilon^2 g_\Sigma \oplus g_{S^1}, a, f)$

equations we shall introduce below. The tangent space of X in $(A, \omega) \in X$ is $T_{(A, \omega)}X = \ker L_{(A, \omega)}$, where

$$\begin{aligned} L_{(A, \omega)} : \Omega^1(Y, \text{ad}(P)) \times \Omega^1(Y, \text{ad}(P)) &\rightarrow \Omega^0(Y, \text{ad}(P)), \\ (\alpha, v) &\mapsto d_A^* v + [* \omega \wedge \alpha]. \end{aligned}$$

The formal adjoint of $L_{(A, \omega)}$ is the operator

$$L_{(A, \omega)}^* : \varphi \mapsto ([\varphi \wedge \omega], d_A \varphi).$$

The orthogonal projection $\pi_{(A, \omega)} : T_{(A, \omega)}(\mathcal{A}(P) \times \Omega^1(Y, \text{ad}(P))) \rightarrow T_{(A, \omega)}X$ is given by

$$\pi_{(A, \omega)}(\alpha, v) = (\alpha - [\varphi, \omega], v - d_A \varphi),$$

where φ solves the elliptic equation

$$(45) \quad R_{(A, \omega)} \varphi := \Delta_A \varphi - [* \omega \wedge [\omega \wedge \varphi]] = L_{(A, \omega)}(\alpha, v).$$

Note that the operator $R_{(A, \omega)}$ is symmetric. For $(\alpha, v) = (*d_A \omega, -\omega + *F_A)$ it follows (assuming $d_A^* \omega = 0$) that

$$(46) \quad R_{(A, \omega)} \varphi = -[*d_A \omega \wedge * \omega] = -\frac{1}{2} * d_A [\omega \wedge \omega].$$

Remark 16. For $(A, \omega) \in X$ where $R_{(A, \omega)}$ is not bijective, it follows that φ in (45) is not uniquely defined. To single out a particular solution it might be useful to require that φ is L^2 orthogonal to the (finite-dimensional) affine subspace of solutions of (45). This leads to the further condition $\varphi \in \text{im } R$.

Definition 17. The *restricted L^2 gradient flow* is the system of equations

$$(47) \quad \begin{cases} 0 = \partial_s A - d_A \Phi + *d_A \omega + [\varphi \wedge \omega] \\ 0 = \partial_s \omega + [\Phi, \omega] - \omega + *F_A + d_A \varphi \end{cases}$$

for a pair $(A, \omega) \in X$ and $\Phi \in \Omega^0(\Sigma, \text{ad}(P))$. Here φ is defined to be a solution of (46).

We conclude this section with a number of remarks. First one should note that the additional condition $d_A^* \omega = 0$ imposed on the (infinite-dimensional) manifold of pairs (A, ω) is compatible with the critical point equation (6). Namely then, $\omega = *F_A$ and hence $d_A^* \omega = 0$ holds by the Bianchi identity. Second, the linearization of (47) together with a gauge fixing condition as discussed in Remark 3 now leads to an elliptic system in any dimension n (in contrast to the unrestricted flow (5), cf. Proposition 15). A slight complication now comes from the fact that the term φ in (47) is nonlocal (however of lower order). Still it is conceivable that an analysis of moduli spaces of solutions of (47) is possible in analogy to that of (5). This will lead to an elliptic Yang–Mills Morse homology for manifolds of dimension $n \geq 3$. We leave this programme to be carried out to a future publication.

APPENDIX A. A PRIORI ESTIMATES

A.1. Differential inequalities for the energy density. From the flow equation (5) it follows by differentiation with respect to s that

$$(48) \quad \begin{cases} \ddot{A} &= d_A^* F_A + (-1)^n * d_A \omega + (-1)^n * [\dot{A} \wedge \omega] \\ \ddot{\omega} &= d_A^* d_A \omega + \dot{\omega} \end{cases}$$

We restrict attention to the case $n = 2$ where (48) constitutes an elliptic system of equations for (A, ω) . Our aim is to derive differential inequalities satisfied by the energy density

$$e(A, \omega) := \frac{1}{2} (|d_A \omega|^2 - |*F_A - \omega|^2).$$

associated with a solution of (5). We denote

$$\Delta_{I \times Y} := -\frac{d^2}{ds^2} + \Delta_Y.$$

Then it follows that

$$\begin{aligned} \Delta_{I \times Y} e &= -\left| \frac{d}{ds} d_A \omega \right|^2 - \left| \frac{d}{ds} (-\omega + *F_A) \right|^2 - |\nabla_A d_A \omega|^2 - |\nabla_A (-\omega + *F_A)|^2 \\ &\quad + \underbrace{\langle d_A \omega, (-\frac{d^2}{ds^2} + \nabla_A^* \nabla_A) d_A \omega \rangle}_I \\ &\quad + \underbrace{\langle -\omega + *F_A, (-\frac{d^2}{ds^2} + \nabla_A^* \nabla_A) (-\omega + *F_A) \rangle}_II. \end{aligned}$$

Now we consider the last two terms. We replace $\nabla_A^* \nabla_A$ using the Weizenböck formula. This results in the expressions $(-\frac{d^2}{ds^2} + \Delta_A) d_A \omega$ and $(-\frac{d^2}{ds^2} + \Delta_A) (-\omega + *F_A)$ for which we obtain from (48) the identities

$$\begin{aligned} &(-\frac{d^2}{ds^2} + \Delta_A) d_A \omega \\ &= -\frac{d}{ds} (d_A \dot{\omega} + [\dot{A} \wedge \omega]) + d_A d_A^* d_A \omega + d_A^* d_A d_A \omega \\ &= -d_A \ddot{\omega} - *[\dot{A} \wedge \dot{\omega}] - [\ddot{A} \wedge \omega] - [\dot{A} \wedge \dot{\omega}] + d_A d_A^* d_A \omega + d_A^* [F_A \wedge \omega] \\ &= -d_A \dot{\omega} + d_A \frac{d}{ds} *F_A - 2[\dot{A} \wedge \dot{\omega}] - [\ddot{A} \wedge \omega] + d_A^* [F_A \wedge \omega] + d_A d_A^* d_A \omega \\ &= -d_A \dot{\omega} + d_A * d_A \dot{A} - 2[\dot{A} \wedge \dot{\omega}] - [\ddot{A} \wedge \omega] + d_A^* [F_A \wedge \omega] + d_A d_A^* d_A \omega \\ &= -d_A \dot{\omega} - 2[\dot{A} \wedge \dot{\omega}] - [\ddot{A} \wedge \omega] + d_A^* [F_A \wedge \omega], \end{aligned}$$

and

$$\begin{aligned}
\left(-\frac{d^2}{ds^2} + \Delta_A\right)(-\omega + *F_A) &= \ddot{\omega} - * \frac{d}{ds} d_A \dot{A} - \Delta_A \omega + d_A^* d_A * F_A \\
&= \ddot{\omega} - * d_A \ddot{A} - *[\dot{A} \wedge \dot{A}] - \Delta_A \omega + d_A^* d_A * F_A \\
&= 2d_A^* d_A \omega - \Delta_A \omega + \dot{\omega} + d_A^*[\dot{A} \wedge \omega] - *[\dot{A} \wedge \dot{A}].
\end{aligned}$$

Because $d_A^* \omega = 0$, the last expression reduces to

$$\left(-\frac{d^2}{ds^2} + \Delta_A\right)(-\omega + *F_A) = \Delta_A \omega + \dot{\omega} + d_A^*[\dot{A} \wedge \omega] - *[\dot{A} \wedge \dot{A}].$$

It then follows (for any $\varepsilon > 0$ and some absolute constant $C > 0$) that

$$\begin{aligned}
|\text{I}| &\leq |\langle d_A \omega, \{F_A, d_A \omega\} + \{R_Y, d_A \omega\} \rangle| \\
&\quad + |\langle d_A \omega, -d_A \dot{\omega} - 2[\dot{A} \wedge \dot{\omega}] - [\ddot{A} \wedge \omega] + d_A^*[F_A \wedge \omega] \rangle| \\
&\leq C(|d_A \omega|^3 + |F_A|^3 + |d_A \omega|^2 + \varepsilon |d_A \dot{\omega}|^2 + \varepsilon^{-1} |d_A \omega|^2 + |\dot{A}|^3 + |\dot{\omega}|^3 \\
&\quad + \varepsilon |\dot{A}|^2 + \varepsilon^{-1} |\omega|^6 + \varepsilon |\nabla_A F_A|^2 + |F_A|^3 + |\nabla_A \omega|^3)
\end{aligned}$$

and

$$\begin{aligned}
|\text{II}| &\leq |\langle -\omega + *F_A, \{F_A, -\omega + *F_A\} + \{R_Y, -\omega + *F_A\} \rangle| \\
&\quad + |\langle -\omega + *F_A, \Delta_A \omega + \dot{\omega} + d_A^*[\dot{A} \wedge \omega] - *[\dot{A} \wedge \dot{A}] \rangle| \\
&\leq C(|-\omega + *F_A|^3 + |F_A|^3 + |-\omega + *F_A|^2 \\
&\quad + \varepsilon |\nabla_A d_A \omega|^2 + \varepsilon^{-1} |-\omega + *F_A|^2 + |\dot{\omega}|^2 + \varepsilon |\nabla_A \dot{A}|^2 \\
&\quad + \varepsilon^{-1} |-\omega + *F_A|^3 + \varepsilon^{-1} |\omega|^6 + |\dot{A}|^3 + |\nabla_A \omega|^3).
\end{aligned}$$

Now we fix $\varepsilon < C^{-1}$. Then the next lemma is a consequence of the last two estimates.

Lemma 18. *Assume that (A, ω) is a solution of (5). Then the energy density $e(A, \omega)$ satisfies on $I \times Y$ the pointwise estimate*

$$(49) \quad \Delta_{I \times Y} e \leq A_0 + A_1 |\omega|^6 + A_2 e + A_3 e^{\frac{3}{2}}$$

for positive constants A_0, A_1, A_2, A_3 which do not depend on (A, ω) .

Remark 19. Note that the exponent $\frac{3}{2}$ appearing in the differential inequality for e in Lemma 18 is critical in the case $\dim Y = 3$.

Lemma 18 together with Theorem 20 below implies an L^∞ bound for e on $I \times Y$ in terms of

$$\int_{I \times Y} e \quad \text{and} \quad \sup_{x \in I \times Y} |\omega(x)|^6 \leq 1 + \|\omega\|_{L^\infty(I \times Y)}^6.$$

As we explain next, the term $\|\omega\|_{L^\infty(I \times Y)}^6$ can be absorbed in the left-hand side of that estimate. Namely, by the assumption that the energy $\int_I e(s) ds$

be bounded and Lemma 21 it follows that there is a constant $C(I) > 0$ such that

$$\|F_{\mathbb{A}}\|_{L^2(I \times Y)} \leq C(I),$$

where we let $F_{\mathbb{A}} = F_A + \dot{A} \wedge ds$ denote the curvature of A , viewed as a connection in $\mathcal{A}(I \times P)$. Let $A_0 \in \mathcal{A}(P)$ be a smooth reference connection. Uhlenbeck's weak compactness theorem (cf. [29, Theorem A]) then yields a further constant $C_1(I)$ such that

$$\|A - A_0\|_{W^{1,2}(I \times \Sigma)} \leq C_1(I).$$

Set $\alpha := A - A_0$. Then by Eq. (7) ω satisfies the elliptic equation

$$\ddot{\omega} - \dot{\omega} - \Delta_{A_0}\omega = d_A^*[\alpha \wedge \omega] - *[\alpha \wedge *d_A\omega].$$

By elliptic regularity we obtain the estimate

$$(50) \quad \|\omega\|_{W^{2,2}(I \times Y)} \leq c(\|\omega\|_{L^2(I \times Y)} + \|d_A^*[\alpha \wedge \omega]\|_{L^2(I \times Y)} + \|[\alpha \wedge *d_A\omega]\|_{L^2(I \times Y)}).$$

Using Hölder's inequality and the Sobolev embeddings $W^{1,2}(I \times \Sigma) \hookrightarrow L^p(I \times \Sigma)$ for $1 < p < 6$ and $W^{2,2}(I \times \Sigma) \hookrightarrow L^\infty(I \times \Sigma)$, the last two terms on the right-hand side can further be estimated as

$$\begin{aligned} \|d_A^*[\alpha \wedge \omega]\|_{L^2(I \times Y)} &= \|[d_A^*\alpha \wedge \omega] + *[\alpha \wedge d_A\omega]\|_{L^2(I \times Y)} \\ &\leq c(\|d_A^*\alpha\|_{L^2(I \times Y)}\|\omega\|_{L^\infty(I \times Y)} + \|\alpha\|_{L^4(I \times Y)}\|\omega\|_{L^4(I \times Y)}) \\ &\leq c(\varepsilon\|\omega\|_{W^{2,2}(I \times \Sigma)} + \varepsilon^{-1}\|\alpha\|_{W^{1,2}(I \times \Sigma)}). \end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small, the term $c\varepsilon\|\omega\|_{W^{2,2}(I \times \Sigma)}$ can be absorbed in the left-hand side of (50). The term $\|[\alpha \wedge *d_A\omega]\|_{L^2(I \times Y)}$ can be treated similarly. This finally yields a constant $C_2(I)$ such that

$$(51) \quad \|\omega\|_{L^\infty(I \times Y)} \leq c\|\omega\|_{W^{2,2}(I \times Y)} \leq C_2(I).$$

Therefore estimate (49) in Lemma 18 can be improved to

$$(52) \quad \Delta_{I \times Y} e \leq A_0 + A_2 e + A_3 e^{\frac{3}{2}}$$

for positive constants A_0, A_2, A_3 which do not depend on (A, ω) .

Theorem 20 (Elliptic mean value inequality). *For every $n \in \mathbb{N}$ there exists constants $C, \mu > 0$, and $\delta > 0$ such that the following holds for all metrics g on \mathbb{R}^n such that $\|g - \mathbb{1}\|_{W^{1,\infty}} < \delta$. Let $B_r(0) \subseteq \mathbb{R}^n$ be the geodesic ball of radius $0 < r \leq 1$. Suppose that the nonnegative function $e \in C^2(B_r(0), [0, \infty))$ satisfies for some $A_0, A_1, a \geq 0$*

$$\Delta e \leq A_0 + A_1 e + a e^{(n+2)/n} \quad \text{and} \quad \int_{B_r(0)} e \leq \mu a^{-n/2}.$$

Then

$$e(0) \leq C A_0 r^2 + C(A_1^{n/2} + r^{-n}) \int_{B_r(0)} e.$$

Proof. For a proof we refer to [30, Theorem 1.1]. \square

A.2. L^2 -estimates. For simplicity, we let $\Phi = 0$ in the following discussion. Let (A, ω) be a solution of (5) such that for critical points (A^\pm, ω^\pm) the asymptotic conditions

$$\lim_{s \rightarrow \pm\infty} (A(s), \omega(s)) = (A^\pm, \omega^\pm)$$

are satisfied. Let $C^\pm := \mathcal{J}(A^\pm, \omega^\pm)$. Since (A, ω) is an L^2 -gradient flow line of \mathcal{J} it follows that

$$(53) \quad \int_{-\infty}^{\infty} \|\nabla \mathcal{J}(A(s), \omega(s))\|_{L^2}^2 ds = C^- - C^+ \geq 0.$$

From this we obtain the following estimate for the L^2 -norm of F_A over the domain $I \times Y$.

Lemma 21. *For (A, ω) as above and any interval $I \in \mathbb{R}$ there holds the estimate*

$$\int_I \|F_{A(s)}\|_{L^2}^2 ds \leq C^- - C^+ + 2 \int_I \mathcal{J}(A(s), \omega(s)) ds \leq (1 + 2|I|)C^- - C^+.$$

Proof. From the energy identity (53) it follows that

$$\begin{aligned} C^- - C^+ &\geq \int_I \|\omega - F_A\|_{L^2}^2 ds = \int_I \|\omega\|_{L^2}^2 + \|F_A\|_{L^2}^2 - 2\langle F_A, \omega \rangle ds \\ &= \int_I \|F_A\|_{L^2}^2 - 2\mathcal{J}(A, \omega) ds. \end{aligned}$$

Note that for all $s \in \mathbb{R}$, $\mathcal{J}(A(s), \omega(s)) \leq C^-$ to conclude the claim. \square

An estimate for the L^2 -norm of ω is given by the following lemma.

Lemma 22. *For (A, ω) as above and every $s \in \mathbb{R}$ there holds the estimate*

$$\frac{1}{2} \|\omega(s)\|_{L^2}^2 \leq \|F_{A(s)}\|_{L^2}^2.$$

Proof. Note that $\omega^+ = F_{A^+}$, hence

$$C^+ = \mathcal{J}(A^+, \omega^+) = \frac{1}{2} \int_Y |F_{A^+}|^2 \geq 0.$$

It therefore follows from the gradient flow property that for all $s \in \mathbb{R}$

$$\begin{aligned} 0 &\leq \mathcal{J}(A(s), \omega(s)) = \int_Y \langle F_{A(s)}, \omega(s) \rangle - \frac{1}{2} |\omega(s)|^2 \\ &\leq \|\omega(s)\|_{L^2} \|F_{A(s)}\|_{L^2} - \frac{1}{2} \|\omega\|_{L^2}^2 = \|\omega(s)\|_{L^2} (\|F_{A(s)}\|_{L^2} - \frac{1}{2} \|\omega(s)\|_{L^2}). \end{aligned}$$

The second line is by the Cauchy-Schwarz inequality. Thus $0 \leq \|F_{A(s)}\|_{L^2} - \frac{1}{2} \|\omega\|_{L^2}$, as claimed. \square

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